

STATISTICS OF N-BODY SIMULATIONS. II. EQUAL MASSES AFTER CORE COLLAPSE

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ABSTRACT

This paper presents and analyses statistical results from a large number of N -body simulations of isolated systems with equal masses, in which $250 \leq N \leq 2000$. It concentrates on the phase starting around the end of core collapse. Binaries play a crucial role, and we find that the total energy of bound pairs is in line with theoretical expectations. Interpretation of the total *number* is complicated by the presence of a number of binaries on the hard/soft threshold. Interactions of hard binaries are consistent with the Spitzer (1987) cross section.

The spatial evolution of the half-mass radius after core collapse nearly follows classical theory, and, by comparison with Fokker-Planck and gas models, allows a redetermination of the effective thermal conductivity and the argument of the Coulomb logarithm in the expression for the relaxation time. The evolution of the inner parts of the system around the time of core bounce is consistent with these simplified models provided that the continuous production of energy, as usually assumed, is replaced by a model of *stochastic* energy production. Similarly, post-collapse evolution of the core requires a modest recalibration of the coefficient of energy generation, especially for small N . These remarks refer to the behaviour averaged over many models; individual cases show alternate and irregular phases of expansion and recollapse. The distributions of velocity dispersion and anisotropy become remarkably homologous soon after core bounce.

The bound mass of the systems very nearly follows a power law with time. A small number of escapers, presumed to be those associated with binary activity, dominate the energy which is carried off: the distribution of energies of escapers changes abruptly at the end of core collapse. The “internal” energy of escaping binaries is consistent with theoretical expectations, and again supports Spitzer’s reaction cross section for hard binaries.

Key words: celestial mechanics, stellar dynamics - globular clusters: general.

1. INTRODUCTION

1.1 The Statistical Study of Idealised N -body Models

The ultimate purpose of this series of papers is the improved understanding of the evolution of star clusters. In this context the study of isolated N -body systems in which all stars have the same mass may seem unrealistic. And yet the dynamical study of real star clusters leans heavily on the theory of several complicated processes, such as the evolution of binary stars, stellar escape, the generation of anisotropy, and so on. What we show in this series of papers is that a careful study of many relatively small and idealised N -body models allows one to test and, in some cases, improve this theory.

Our strategic approach to these questions was discussed in Paper I (Giersz & Heggie 1994). There we argued that the combined study of large numbers of N -body models, which may be done rather easily with parallel computers, leads to results of sufficiently high statistical quality that several old problems of stellar dynamics can be reexamined empirically, leading in turn to new developments in theory. An example from the previous paper is the study of the escape rate, which showed the crucial role of the development of anisotropy, as well as that of the spatial evolution of the system.

Paper I described results up to a point late in core collapse, and the present paper takes up the analysis of our data at that point. One reason for dividing the discussion there is that many aspects of the subsequent evolution of the systems are heavily influenced by the formation and evolution of binary stars. This topic takes up Section 2, and there follow two sections dealing, respectively, with the evolution of the entire spatial structure and the statistical properties of the velocities of the stars. Sections 5 and 6 then discuss in some detail conditions in the core of the systems, and aspects of the escape of stars and binaries, respectively. The final section sums up.

Further papers in this series will extend our approach towards progressively more realistic models for star clusters. Part of our purpose here is to examine the conclusions of the survey, using the isotropic Fokker-Planck model, by Chernoff & Weinberg (1990). It has proved to be one of the most valuable resources for understanding the evolution of globular clusters. Thus our next paper discusses the effects of a spectrum of stellar masses similar to that chosen by them. Later papers consider the influence of tidal effects and stellar evolution (modeled as instantaneous mass loss).

1.2 The Models

In this paper we make use of the same sets of models, with $N = 250, 500, 1000$ and 2000 , as in Paper I, which gives all necessary details of initial conditions, hardware and software considerations, and the kind of data which is available for study. We merely recapitulate the units used, in which the initial total mass and energy are $M = 1$ and $E = -1/4$, respectively, and the constant of gravitation $G = 1$. We refer to these hereafter as “ N -body units”. There is, however, one issue which was irrelevant for Paper I, which is the completeness of the samples.

It was our intention to study models for a time comparable with at least two core collapse times, which implies a time approximately proportional to N . Because the computational effort grows rapidly with N , a compromise was reached in which we attempted to integrate models up to a time t_{end} given (in N -body units) in Table 1. Even so, some

models were lost before time t_{end} because of hardware or software problems, and so for each series Table 1 shows how many models were running at the final time. Even those models which have been successfully completed may be at rather different evolutionary stages at the finishing time: different N -body models, drawn from the same initial conditions except for the choice of random numbers, evolve at surprisingly different rates.

There is little doubt that the time at which some models are lost (because of software problems) is closely linked with their dynamical behaviour, e.g. the prevalence of time-consuming few-body interactions. Coupled with the substantial spread of evolution rates, this can in principle seriously bias the statistical results, and should strictly be avoided. In some of our results, however, we do display data extending to a time at which a few of the models have already stopped, and the reader should bear Table 1 in mind when conclusions are drawn from our results.

2. BINARIES

Many aspects of the evolution of N -body systems, from around the time of core bounce, are dominated by binaries. Examples include the radius of the core and the energy of escapers. One of our aims in this paper is to characterise the evolution of the binaries in terms of relatively simple models, so that the application of these models in studies of larger systems can be carried out with greater confidence. There are three broad classes of models: (i) simple scaling arguments, as in §2.1; (ii) simple models for binary evolution, such as those which are often incorporated into Fokker-Planck and gas models (cf. eq.(13)); and (iii) rather detailed models on the stochastic evolution of individual binaries, based on appropriate reaction cross sections, as in §6.2. The third group of models can also be incorporated into Fokker-Planck and gas models for the evolution of the cluster (cf. §3.2). Most of these models can be tested at various levels, including (i) statistics on the behaviour of individual binaries (e.g. §2.4); (ii) statistics on their numbers and energy (e.g. §§2.2, 2.3), and (iii) the effect they have on the structure of the cluster (e.g. §3.2). These remarks should give the reader a point of reference when some of the numerous combinations of model and test data are discussed in this and succeeding sections.

2.1 Time of First Formation of Binaries

At one time it was thought that the role of binary stars becomes relatively less important the larger the value of N (Spitzer & Hart 1971). Now it is accepted that, at least in the point-mass approximation, binaries play a crucial role in the post-collapse evolution of systems of all sizes (e.g. Spitzer 1987). However, provided that there are no primordial binaries, binaries play a relatively minor role throughout most of core collapse, especially for large N . The purpose of this section is to quantify this statement on the basis of N -body models with $250 \leq N \leq 2000$.

First we outline a theoretical argument (cf. Inagaki 1984). The rate of formation of hard binaries by 3-body encounters in a system with N stars varies as

$$\dot{N}_b \sim \frac{1}{N t_r}, \quad (1)$$

where t_r is the relaxation time. We shall apply this to the core (replacing N by N_c and t_r by t_{rc} to indicate values appropriate to the core), and assume that N_c varies as a power of t_{rc} during late core collapse. If we set

$$\frac{N_c}{N_c(0)} = \left(\frac{t_{rc}}{t_{rc}(0)} \right)^{\frac{2(3-\alpha)}{6-\alpha}}, \quad (2)$$

where a zero denotes an initial value, and we would have $\alpha = 2.21$ in homologous core collapse in a gaseous model (Lynden-Bell & Eggleton 1980), it follows that

$$\dot{N}_b \sim \frac{1}{t_{rc}(0)N_c(0)} \left(\frac{t_{rc}}{t_{rc}(0)} \right)^{-\frac{3(4-\alpha)}{6-\alpha}}. \quad (3)$$

Then, since t_{rc} varies linearly with time in homologous core collapse, and $N_b(0) = 0$, we may estimate that

$$N_b \sim \frac{1}{N_c(0)} \left(\frac{t_{rc}}{t_{rc}(0)} \right)^{-\frac{2(3-\alpha)}{6-\alpha}} \quad (4)$$

by the time $t_{rc} \ll t_{rc}(0)$. Therefore the first binary should form when

$$\frac{t_{rc}}{t_{rc}(0)} \sim N_c(0)^{-\frac{6-\alpha}{2(3-\alpha)}}. \quad (5)$$

Within our approximations this can be read as the fraction of the collapse time which remains when the first binary forms. It is a steep power of $N_c(0) \propto N$, varying as $N^{-2.4}$ approximately for homologous collapse. Therefore the larger the system the more abrupt is the appearance of the first hard binary late in core collapse.

This is a statistical result, and we have found that individual N -body systems vary widely, especially when N is not very large. The empirical results also depend on the definition adopted for a “hard” binary. If $3kT/2$ is the initial mean kinetic energy of single stars, binaries with an energy exceeding $1kT$ are present in a fraction of systems initially when $N \leq 500$, and for $N = 250$ a few systems initially have binaries with energies exceeding $3kT$. Table 2 indicates, for each N and for three different binding energies the times at which half our systems contained a binary with energy above the threshold. It is evident that our largest and smallest systems (with $N = 2000$ and 250 , respectively) represent opposite extremes of behaviour. For $N = 250$ the formation and evolution of binaries proceeds for much of the core collapse phase, whereas for $N = 2000$ they are present only during the last 10% or so of its duration. Indeed this result implies that our estimate of the N -dependence in eq.(5) cannot apply for values of $N \lesssim 250$.

In principle a more detailed comparison between theory and N -body data can be carried out by computing the rate of formation of binaries in an evolving gas or Fokker-Planck model. For this purpose we chose values of the free parameters in these models (a

coefficient in the Coulomb logarithm and, for the gas model, a coefficient in the thermal conductivity) as in Paper I. For the rate of formation of binaries we used the result of Hut (1985). The comparison is not at all straightforward, as Hut’s formula is intended to yield the rate of formation of “permanent” hard binaries, i.e. those which are not subsequently disrupted in encounters (though they may be ejected from the system). This is difficult to find for our N -body models. Goodman & Hut (1993) found that the median energy of a newly formed permanent binary is $2.9kT$. Even so, we have found that the theoretical time of formation of the first permanent hard binary (thus defined) is no earlier than the time t_{10} in Table 1, and indeed still later for the larger values of N . It is possible that the time-averaged rate of formation of binaries is larger than predicted theoretically because of fluctuations in the density of the core which are not modelled in the continuum models.

2.2 Number of Binaries

We now discuss one aspect of the post-collapse phase of the evolution, for the first time in these papers. Following Hénon (1965), theory requires that the overall expansion of the cluster is powered by the evolution of binaries. Since the expansion takes place on the time scale of the half-mass relaxation time t_{rh} , the required power can be estimated as

$$\dot{E} \sim \frac{|E|}{t_{rh}}, \quad (6)$$

where E is the total “external” energy of the system, i.e. excluding the internal binding energy of binaries. The energy yielded by each binary may be assumed to be proportional to the central potential ϕ_c (Goodman 1984, 1987, Statler *et al* 1987), and so the formation rate must vary as $\dot{N}_b \propto E/(m\phi_c t_{rh})$, where m is an individual stellar mass. Assuming that the part of the cluster between the core- and half-mass radii is nearly isothermal, it follows that

$$\dot{N}_b \propto \frac{N\sigma_c^2}{\phi_c t_{rh}}, \quad (7)$$

where σ_c is the root mean square one-dimensional velocity dispersion in the core. Applying eq.(1) to the core we deduce that

$$N_c \propto \left(\frac{N|\phi_c|}{\sigma_c^2} \right)^{1/3}, \quad (8)$$

which we discuss in §5.2.

This argument about the formation rate says nothing about the average numbers of binaries, which depends also on their individual lifetime within the cluster, t_b say. If each binary stayed within the core until ejection we could estimate that its hardening rate is $\sim m\sigma_c^2/t_{rc}$ (Heggie 1975, Hills 1975, though this formula is valid only if we neglect variations in the Coulomb logarithm). Hence we could estimate that

$$t_b \propto t_{rc} \left(\frac{|\phi_c|}{\sigma_c^2} \right). \quad (9)$$

If we do not assume that binaries remain within the core then we can still express our result as

$$N_b \sim \dot{N}_b t_b \propto \left(\frac{t_b \sigma_c^2}{|\phi_c| t_{rc}} \right) \left(\frac{t_{rc}}{t_{rh}} \right) N \quad (10)$$

by eqs.(7) and (9). Using the isothermal approximation again, and eq.(8), it follows that

$$N_b \propto \left(\frac{t_b \sigma_c^2}{|\phi_c| t_{rc}} \right) \left(\frac{|\phi_c|}{\sigma_c^2} \right)^{2/3} N^{-1/3}. \quad (11)$$

The empirical data from our N -body models is somewhat at variance with the simplest interpretation of this formula, i.e. $N_b \propto N^{-1/3}$, since we find in the post-collapse regime that the average number of binaries (defined here as regularised binaries (Aarseth 1985)) actually tends to *increase* with N , but quite weakly, and with large fluctuations (Fig.1). The N -dependence is roughly $N_b \propto N^{0.2}$. A simple possible explanation of this discrepancy, and one for which evidence is presented in the next section, is that our count of binaries includes temporary pairs which contribute nothing to the energetics of the expansion, but whose numbers disguise the smaller number of active binaries. It is possible, however, that t_b much exceeds the estimate given by eq.(9). As a binary hardens it tends to be ejected ever further from the core (Hut *et al* 1992), and dynamical friction acting on a time scale more comparable with t_{rh} than with t_{rc} may determine its lifetime. This would lead roughly to $N_b \propto N$, which differs from the empirical data in the opposite sense but can be ruled out even more strongly. It is quite possible, therefore, that the data of Fig.1 are explicable if the regularised binaries consist of some temporary binaries with lifetimes (to destruction) of order t_{rc} , and others whose lifetimes (to ejection) are of order t_{rh} .

These theoretical comparisons are based on very simple scaling arguments. It is also possible to compute the numbers of binaries that would be formed in gas- and Fokker-Planck models of star clusters. We have modeled the formation and evolution of binaries in these models in several different ways (cf. §3.2 below), and all predict that the formation of binaries during late core collapse occurs too slowly in comparison with N -body data (cf. §2.1). Here, however, we are concerned with the predicted numbers after core collapse.

The standard way of treating these processes in gas and Fokker-Planck models is to assume that binaries form at a smooth rate given by a refined and local version of eq.(1), and that they instantly emit an amount of energy of order $|\phi_c|$. Therefore these models cannot be used to predict the current number of binaries bound to the system. The simplest basis for comparison with the N -body data is to compute the total number of binaries formed in the continuum model and to compare this with N -body data on the sum of all regularised bound binaries plus all regularised binaries which have been removed from the system. Even so, the predicted numbers of binaries are smaller than those obtained from the N -body models, and the discrepancy is bigger for larger N .

Gas models which treat the formation of binaries and their subsequent burning as a *stochastic* process can reproduce a nearly constant number of bound binaries after core

collapse, as found in N -body models (see Fig.1). For different models the number levels off at different values, which depend on the assumptions about how much of the energy generated by binaries is directly supplied to the core, and on the form of the cross section for binary hardening (cf. §2.4 below). We did not, however, attempt to evaluate the stochastic gas models by comparing the empirical and predicted numbers of binaries, because the empirical (N -body) data are contaminated in at least in two ways. First, as was mentioned in §2.1, it is very difficult to distinguish between permanent and non permanent binaries for N -body models. Second, hard binaries can spend a substantial time in the halo, because interactions with field stars remove them from the core, and that process was not simulated in our gas models. Therefore we can expect that the “real” number of binaries which contribute to the cluster energy balance (which is appropriate for comparison with the gas models) should be smaller (by a factor which may be of order unity) than the value obtained from N -body data. The N -dependence of the number of binaries obtained from stochastic gas models is similar to that predicted by eq.(11), (with t_b estimated as in eq.(9)), in the sense that the number of binaries is smaller for larger systems, but the power-law index is slightly smaller. The average value over all models is about -0.13 , whereas the predicted value is about -0.2 when we include the observed N -dependence of the scaled central potential ϕ_c/σ_c^2 (cf. §5.1 and Fig.9).

A hybrid way of testing the theoretical binary formation rate is to take the N -body data on the spatial structure and the velocity dispersion profile, and to apply to it the standard formula (Hut 1985) for the local rate of formation of permanent hard pairs. Just as for the theoretical production of hard binaries in gas models (§2.1), this predicts a population of binaries that increases too slowly in the collapse phase. In the post-collapse phase, however, it produces generally (except for $N = 250$) somewhat too few binaries (i.e. by comparison with the total number of bound and unbound regularised binaries in the N -body data). A very clear trend with N can be observed, in the sense that the discrepancy between the actual and predicted numbers of binaries increases with N . This trend can be at least partially explained by the way in which the predicted number of binaries is estimated from the N -body data. The rate of binary formation is a strong function of density, which in turn is mainly determined by the innermost Lagrangian radii. For decreasing N the number of stars in each Lagrangian shell decreases, and so the density estimation is poorer and has larger fluctuations. Positive density fluctuations bias the predicted number of binaries to high values, and this bias is greater for smaller systems.

Most of the remarks in this section refer to the early post-collapse phase. For systems in which we can observe the long term post-collapse evolution ($N = 250$ and 500) the estimated and computed rates of binary formation become very similar. In a sense this finding supports H  non’s theory that, in post-collapse evolution, energy generated by binaries is adjusted to the energy demanded by the overall expansion. It appears that most of the discrepancies between the theoretical and N -body models arise near core bounce.

2.3 Energy of Binaries

We now consider the internal energies of hard binaries which are bound to the system. The lower limit for an individual binary is set by the definition of ‘hard’, which is often taken to mean energies above $1kT$. The upper limit is set by the escape of binaries following

energetic interactions: the mean energy imparted to the barycentric motion of the binary is a certain fraction of its internal energy, and so the binary is likely to escape after an energetic interaction if its internal binding energy exceeds a certain multiple of the escape energy.

Because of the stochastic aspects of such interactions, it is not surprising that there are wide variations in the maximum binding energy of binaries in our N -body models; the maximum occasionally reaches values comparable with the entire binding energy of the system. The *average* (over all cases at a given time) of the maximum individual binding energy of a binary is quite well behaved, however (Fig.2). Models for all N show a slow decline during the post-collapse phase, and this decline is consistent with supposing that the maximum is approximately proportional to the escape energy from the centre (Fig.9 and §6.3).

When we turn to the *total* internal energy of bound binaries (which we refer to as $E_{b,int}$) there is a trend with N which is more consistent with theoretical expectations than is the case for the observed *number* of binaries. At comparable times in the post-collapse phase the total energy is smaller for larger systems. The maximum occurs early in the post-collapse phase, and may be estimated very roughly as $E_{b,int} \simeq 0.06(N/1000)^{-1/3}$; the power is suggested by the arguments of §2.2 (i.e. eq.(11), and assuming that the energy of each binary is proportional to σ_c^2 or ϕ_c , and that $t_b \sim t_{rc}$), and the coefficient is determined from the N -body data. The fact that the N -dependence is roughly consistent suggests that the total energy is almost entirely contributed by at most a single hard binary, whereas the balance of the observed numbers of binaries is made up by much less energetic pairs which are almost always present after core collapse.

Besides this roughly quantified N -dependence at comparable times, there is a noticeable decline with time in the total energy of binaries. This nearly follows a similar trend, noted above, in the maximum energy of all binaries.

It was noted in §2.1 that the first hard binaries tend to form more quickly than one would expect in comparison with theoretical results on the formation rate of “permanent” hard pairs. The opposite effect occurs (i.e. binaries are less effective than expected) if one computes the energy generated by binaries. In a gas model this is often estimated using the following rather standard formula for the rate of generation of energy per unit mass:

$$\varepsilon = 7.5G^5 f_2 f_3 m^3 \left(\frac{|\phi_c|}{\sigma_c^2} \right) \rho^2 \sigma^{-7}, \quad (12)$$

(cf. Goodman 1987), where m is the mass of an individual star, ρ is the density, σ the one-dimensional velocity dispersion, and f_2 and f_3 are coefficients (of order unity) which determine the rate of binary formation and how much energy released by binaries goes directly to the core, respectively. In later discussion we shall sometimes refer to the entire dimensionless part of the coefficient as C_b , i.e.

$$\varepsilon = C_b G^5 m^3 \rho^2 \sigma^{-7}. \quad (13)$$

A value of $C_b = 90$ is often used (Cohn 1985). In our N -body systems the energy in hard binaries catches up with the prediction of this formula late in core collapse, but up

until then the predicted energy generation is too large. We consider that the cause of this discrepancy is the assumption, implicit in the formula, of “instantaneous burning”, i.e. that a newly formed binary will instantly yield the energy which, in fact, is spread out over its lifetime. This issue is discussed further in §3.2 below. On the other hand, during the late pre- and early post-collapse phase (around the time of core bounce) the energy generated in N -body systems is much bigger than in the gas models. In other words, the N -body models appear to require energy to be supplied at a faster rate than in the gas models to stop and reverse the core collapse and then establish post-collapse expansion. Subsequently, as with the rate of binary formation, the rate of energy generation during the advanced post-collapse phase is virtually the same for both N -body and gaseous models. What has just been said about models with continuous energy generation is also true for those with stochastic energy generation, but during the collapse phase the stochastic models give much better agreement with the N -body data.

2.4 Interactions of Binaries

Since our data is recorded only once each time interval, and since we have little information on the identity of binaries present at each time, it is difficult to present unambiguous data on such problems as the changes in energy experienced by binaries in interactions. However, since it is unlikely that there is more than one very energetic binary present in each model at each time (§2.3), the results should give a good guide to *its* behaviour. For example, a study of the energy changes (between successive time units) above a suitable threshold indicates an average relative change of order 0.2 or less, though with much scatter, especially for the most energetic pairs. The value 0.2 is predicted by the binary scattering cross section given by Spitzer (1987), which is itself rather close to that found numerically by Heggie & Hut (1993). In our view this value of the mean relative energy change supersedes the value of 0.4 which was originally suggested by scattering experiments at low impact parameter (Hills 1975) and the theory of close interactions (Heggie 1975). The mean relative change in binding energy per interaction is relevant to the energy which a binary reaches before being liable to escape and to the total energy which each hard binary yields before escaping (§6.3).

The scattering cross sections just mentioned predict a steep fall-off with increasing Δ (the relative increase in binding energy), and considerably steeper than found from our data. This may occur because more than one weak encounter may occur within our sampling interval, which suppresses the correct abundance of encounters with low values of Δ . Better statistics on these questions, again made by direct observation of N -body systems, can be found in McMillan (1989).

The larger energy changes are responsible for ejecting binaries to large radii (Hut *et al* 1992). Our results show that there is an increasing *spread* in the spatial distribution of hard binaries with time. While there is almost always at least one binary within the core in the post-collapse phase, the mean distance of the most outlying one increases roughly comparably with the half-mass radius.

Though the binaries may be found over a wide range of radii, their interactions take place when they are in the regions of higher density. Almost all interactions (defined as any recorded change in the binding energy above an arbitrarily chosen threshold, when there was no ambiguity in the position of the binary) take place within the half-mass radius,

and at least half within the core radius.

The above discussions on the spatial distribution of hard binaries within the cluster and the locations of their interactions raise the following question: Where does a binary deposit the energy it emits in an interaction? The answer to this question depends on the relation between the escape velocity from the cluster and the escape velocity from the core. The smaller the system the smaller is this ratio (see Fig.20, which portrays the central potential scaled by the central velocity dispersion), and the easier it is to remove stars and binaries from the core, and to disperse them widely throughout the cluster. This means that for low N the energy generated by binaries is more widely distributed within the cluster than for high N , where it should be relatively more concentrated within the core. This difference is reflected in the spatial evolution of the cluster. The intermediate and outer Lagrangian radii evolve considerably faster for $N = 250$ than for larger models even after allowance is made for the difference in relaxation times (see Giersz & Spurzem 1993 for more discussion). The slopes of the profiles of density and velocity dispersion are also markedly different for models with low and high N (cf. §§3.3, 4.2).

The significance of this discussion is that any formula of the type presented in eq.(12) cannot correctly represent the spatial and temporal distribution of energy generation in N -body models, though results in later sections of this paper suggest that it can be treated as a reasonable compromise between the real complexity of binary interactions in N -body systems and the need for a simple formula which can be used in gas or Fokker-Plank models.

3. SPATIAL EVOLUTION

3.1 The Half-Mass Radius

As mentioned in §2.2, orthodox theory implies that the post-collapse evolution of the total energy of a system is determined by average properties of the whole cluster, and not by dynamical processes in the core. We follow this distinction by discussing separately the evolution of the inner Lagrangian radii (those which contract during core collapse) and the outer parts of a cluster. For the latter we concentrate on the half-mass radius.

Studying the N -dependence of this evolution is complicated by the fact that our models reach rather different stages of post-collapse evolution, in a rather N -dependent way. Our clusters for $N = 250$ are virtually complete for about 11 collapse times, whereas for $N = 1000$ they stop at about 2 collapse times.

For $N = 250$ the late evolution is sufficiently far advanced that loss of mass by escape (cf. §6) has had a significant effect. Let us parameterise this by supposing that the bound mass M varies with t as

$$M(t) = a(t - t_0)^{-\nu}, \quad (14)$$

(Goodman 1984), where t_0 and ν are constants. Assuming that $t - t_0$ is proportional to the half-mass relaxation time, and neglecting the variation with time of the Coulomb logarithm, these simple considerations lead to the result

$$r_h(t) = b(t - t_0)^{(2+\nu)/3}, \quad (15)$$

where b is another constant.

Empirically, by fitting the dependence of the bound mass and half-mass radius on t , we can find values of ν . Unfortunately these values depend on the range of t over which the fit is carried out. We found that by combining eqs.(14) and (15) we can get more consistent results by fitting the half-mass radius as a function of bound mass, i.e. the relation obtained by eliminating t in the above equations. The values of ν for all models are presented in Table 3.

The theoretical form of eqs.(14) and (15) corresponds to the self-similar gas model computed by Goodman (1984) and, if we ignore ν , to the Fokker-Planck model of Hénon (1965). Comparison of the parameter a or b in eqs.(14) and (15) with the corresponding forms in these models permits a determination of the parameters γ in the Coulomb logarithm and also a constant C which appears in the gas model in the coefficient of thermal conductivity (Lynden-Bell & Eggleton 1980). Note that we have chosen a different method of determining these parameters from the method used in the analysis of core collapse in Paper I. There are reasons for supposing that the best values of the parameters γ and C will be different in collapse and post-collapse, and so the latter values cannot be determined by an overall scaling of the results of an evolutionary gas- or Fokker-Planck model (i.e. a single scaling valid for both pre- and post-collapse evolution), which would be the closest analogue to the method adopted in Paper I. The best values we have deduced are given in Table 3 with subscript 1, but note that γ was determined solely by comparison with Hénon's model (taking $\nu = 0$), and then C was determined from Goodman's model using this value of γ . Assuming that Hénon's model could also have the same ν dependence as Goodman's model if allowance were made for mass loss (i.e. a power-law index equal to $(2 + \nu)/3$ for the dependence of the half-mass radius on time, instead of $2/3$) we can estimate new values of γ and C , which are given in Table 3 with subscript 2.

Perhaps, however, the best way of estimating these parameters is to eliminate ν from eqs.(14) and (15) and fit the resulting formula

$$M^{1/2}r_h^{3/2} = c(t - t_0), \quad (16)$$

to the N -body data. The coefficient c is a known function of γ for Hénon's model, and of γ and C for Goodman's model, and so we can determine them using values of c derived for each N . The best overall values deduced in this way are $C = 0.164$ and $\gamma = 0.035$.

The value of C can also be estimated, without any reference to N -body models whatever, by direct comparison between Hénon's and Goodman's models, provided that ν can be set equal to zero in the latter. The result depends on the value of the scaled half-mass radius in Goodman's model, which is a weak function of N . For the limit of low N we found that C is equal 0.158.

The consistency of these values gives an indication of how well the N -dependence of the evolution is captured by these models. Note that the value of C is about 50% larger than the best value found for core collapse (Paper I). However the latter value was determined mainly by analysis of the inner parts of the systems. As was mentioned in Paper I, comparison between gas and Fokker-Planck models indicates that the effective value of C is larger for the outer parts of a system during core collapse. Also, a determination of γ and C for the half-mass radius of N -body models during core collapse suggests values of 0.05 and 0.125, respectively. Therefore the values found for the post-collapse evolution

do not differ so greatly from those obtained for the same Lagrangian radius in the collapse phase. Unfortunately, determination of the best values of C and γ for the inner Lagrangian radii in post collapse evolution could not be carried out in the same way as for the half-mass radius, because we do not know the values of the scaled inner Lagrangian radii for Goodman’s models (Goodman 1984). Nevertheless we believe that the values of C and γ do change in the post-collapse phase. C should be bigger and γ smaller than in the collapse phase, but the appropriate values may depend also on which Lagrangian radius is being considered.

Closely associated with the evolution of the half-mass radius is the total energy of the bound members of the system, if we exclude the internal binding energy of bound binaries. This defines what Aarseth & Heggie (1992) termed the “external” energy of the system, and it is plotted in Fig.3. Here the models for $N = 250, 500, 1000$ and 2000 are plotted together, using the N -dependent scaling deduced in Paper I; though this scaling is based on the behaviour of the collapse phase, whereas we have found a different value for C after core collapse, this does not affect the relative location of the three curves on this plot. What is evident here is, once again, the later production of energy in core collapse for larger N (cf. Fig.2), whereas the production of energy becomes more and more closely synchronised after core collapse, in agreement with Hénon’s argument.

3.2 The Inner Lagrangian Radii

Before we discuss the interpretation of the spatial evolution of the inner parts of the cluster, we present a typical result (Fig.4) which summarises the entire evolution. It is clear visually that the behaviour of the systems is close to being self-similar throughout almost all of the post-collapse expansion, at least as far as the half-mass radius. (The radii containing the innermost 75% and 90% of the mass expand slightly faster, however.) This nearly self-similar evolution is exhibited by our models with $N = 250, 1000$ and 2000 as well. Another way of presenting this data, though it is not shown here, is to compute a logarithmic density profile (based on the mean density between each Lagrangian sphere) plotted against a radial variable scaled to r_h : after core collapse the profiles are virtually identical except for a time-dependent vertical shift.

According to standard theory, in post-collapse expansion the core adjusts itself so that energy is released there at just the rate required to support the expansion of the outer parts of the system. Therefore an adequate theoretical explanation of the behaviour of the inner Lagrangian radii depends on successful modelling of the mechanism of energy generation, which in N -body models is usually thought of as being due to the formation, evolution and expulsion of binaries formed in three-body encounters.

In the first place we compared the evolution of our N -body models with that of a gas model in which binaries yield energy at a rate given by eq.(13). For $N = 250$ this was found to make the minimum of the innermost Lagrangian radii (at “core bounce”) too shallow; a good fit here required a reduction in the coefficient to about 25, for which we could find no plausible explanation.

Note that eq.(13) assumes that binaries are formed at a certain rate, and instantaneously emit a certain amount of energy. Two obvious shortcomings of these assumptions are that both the formation and the evolution of a binary are stochastic. In fact it is not difficult, within the scope of a gas model, to incorporate these effects, as has been done by

Takahashi & Inagaki (1991) for the Fokker-Planck model. For the record we state here the formulae we used to model the stochastic effects. In each time step the formation rate of permanent binaries was computed from the formula of Hut (1985), and a pseudo-random number was used to decide whether a binary forms. If so it is added to a list of existing binaries. In each time step each binary is tested, again probabilistically, to decide if it experiences an encounter with a single star. For this purpose the scattering cross sections of either Heggie (1975, eq.(5.65)) or Spitzer (1987, eq.(6-27)) were used, the location of the binary being assumed to be the centre of the system. In the event of an encounter, a fraction of the energy released is added to the core. (In gas models we cannot follow movements of binaries and single stars in the system, and therefore we do not know where the energy released by binaries is deposited. To overcome this problem we introduced a parameter which describes what fraction of the total energy generated by binaries is deposited in the core. If it is smaller than one the remaining energy is assumed to have no influence on the system evolution.) If the increase in the translational speed of the binary takes it above the escape speed from the centre (assuming that the initial translational speed is $v_c/\sqrt{2}$, where v_c is the central root-mean square three-dimensional velocity), the binary is removed from the list.

We found that stochastic models gave a deeper minimum of the inner Lagrangian radii at the end of core collapse than continuous models. In fact it was rather *too* deep, and a possible explanation for this lies in our technique of averaging results for a number of N -body and gas systems. The spread of times of core bounce for N -body models is bigger than for stochastic gas models due to fluctuations caused by the small numbers of stars in the cores of the N -body models. Because of this spread, the average Lagrangian radii for N -body models will reach shallower minima than stochastic models.

Except for this feature, the stochastic models can be adjusted to provide a very good fit to all properties of the core (core radius, central density and velocity dispersion) for the whole evolution. We found that a stochastic model in which about half of the total energy generated by binaries is deposited in the core, with Spitzer's scattering cross section, gave the best fit to the N -body data. Other models (e.g. Spitzer's scattering cross section with all the energy generated by binaries being deposited in the core, or Heggie's scattering cross section) gave values of core parameters which were systematically in error, especially around core bounce. However, we would like to stress that all stochastic models which we tried gave reasonably good fits for the post-collapse evolution. In this phase the evolution is monotonic, and one might expect that stochastic effects would play a less essential role. Therefore we have also considered how well a gas model based on eq.(13) fits the N -body data. For $N = 250$ the data are fitted best if the value of C_b is reduced to about 55, though higher values, nearer 70 and 90, are preferred for $N = 500$ and $N \geq 1000$, respectively (cf. Giersz & Spurzem 1993).

Several factors contribute to the value of this coefficient, and it is not immediately clear which one has to be modified to account for our empirical finding that, for small N , a smaller value is needed than the one usually adopted (i.e. $C_b = 90$, cf. Cohn 1985). Among these factors are the following: (i) the heating may be less localised than is implicitly assumed in eq.(13); (ii) the total energy emitted by a binary before it escapes depends on the mean energy emitted in each encounter: eq.(13) with $C_b = 90$ is based

on a value 0.4 for the mean relative change in energy (Heggie 1975 and Hills 1975), but we have already seen (§2.4) that there are reasons for supposing this is too high; (iii) the energy emitted per binary is related to the central potential, and in the systems with small N this is smaller (relative to the velocity dispersion) than is usually assumed (see §6). Explanations (i) and (iii), which are related, predict a dependence on N which is consistent with our N -body data. For more discussion on this issue we refer to the paper of Giersz & Spurzem (1993).

Our assertion that stochastic phenomena have little effect after core bounce may apply to average results for an ensemble of systems, but it does not apply to individual cases. Fig.5 shows results for one particularly prolonged calculation. It exhibits a small number of episodes in which the inner Lagrangian radii show abrupt expansion (by as much as a factor of two) followed by slow recollapse. It is customary to ascribe such behaviour to the loss from the core of a binary (by recoil after an energetic interaction), but what puzzles us about results such as this is that relatively few such interactions are accompanied by this kind of abrupt expansion, and those that are are not necessarily the most energetic and spectacular ones.

It is clear that these episodes of collapse and reexpansion (superimposed on the long-term post-collapse expansion) can occur at different times in different systems. For this reason it turns out that the dispersion in the Lagrangian radii (over a given ensemble of models) is much larger in post-collapse evolution than during core collapse. For all Lagrangian radii during the post-collapse phase the dispersion is nearly a constant fraction of radius. These fractions increase with decreasing radius, reaching values of about 0.5 for the innermost Lagrangian radius (in the case $N = 500$).

3.3. Logarithmic Gradient Profiles

At the end of this section we would like to present our results on the spatial evolution of N -body systems in the form of profiles of the logarithmic density *gradient*. One might think that this would be of no value because of the statistical noise in the N -body data. We shall see, however, that this may be overcome by the simple techniques of combining data from large numbers of independent simulations, and time-smoothing. In another sense what we shall be doing is presenting the data of Fig.4 in a different way, and we shall discuss corresponding results for other N .

Our N -body models provide information about the radii of Lagrangian shells, and not information on the density directly. In order to convert the available data into profiles of density gradients, let us assume that the density can be expressed by the function $\rho = Ar^{-\alpha}$ within two successive Lagrangian shells. By using this formula to compute the masses of the two shells (i.e. between three consecutive Lagrangian radii) and comparing with the N -body data, it is possible to estimate the value of α , the logarithmic density gradient, at the middle radius. However, in order to reduce the statistical “noise” of the N -body data further, it was first smoothed using the standard routine SMOOFT from Numerical Recipes (Press *et al* 1986), and then the procedure described above was implemented.

To check the validity of our method we used it to compute the logarithmic density gradient profiles for some isotropic and anisotropic gas models (Spurzem 1993), i.e. profiles based on the Lagrangian radii of these models, and compared them with the values given directly by the models. The results obtained by both methods agreed very well.

Turning now to the profiles obtained from our N -body data, we find that the results vary rather markedly with N (see Fig.6a,b). For $N = 250$ the core collapse does not proceed to sufficient depth to develop the structural characteristics of self-similar gravothermal collapse, i.e. a constant value of α for Lagrangian radii around the collapsing core (Cohn 1980). It seems that core collapse does not enter this self-similar phase at all. The post-collapse profiles agree very well, however, with a homologous expansion scenario, as all profiles are very similar to that at the time of core bounce. By contrast, the characteristic constant value of α is clearly seen in a gas model for $N = 250$ (Fig.6c).

For bigger systems the N -body models develop deeper collapse, and the logarithmic gradient profiles of density become more and more similar to those for gas models. The profiles for $N = 2000$, for both N -body and gas models, are presented in Figs.6b and 6d, respectively. What is evident from Fig.6d is the decrease of α (increase of $d \ln \rho / d \ln r$) at intermediate Lagrangian radii, both for very advanced core collapse and for homologous post-collapse expansion. Similar behaviour for N -body data is clearly visible only for $N = 10000$ (Spurzem 1993). This tendency is qualitatively consistent with the post-collapse homologous models of Inagaki & Lynden-Bell (1983) at least for a limited range of radii. Real systems with smaller N do develop homologous expansion but their spatial structure is more complicated than simple theory predicts.

The technique described above provides useful additional information on the spatial structure of N -body systems, and in future it could be used to examine more realistic N -body calculations of systems with larger N .

4. EVOLUTION OF THE VELOCITY DISTRIBUTION

4.1 Comparison with Continuum Models

With regard to the distribution of stellar velocities, the data discussed in this paper have little to add to what was said in Paper I, until late core collapse. Therefore in this section we concentrate on the velocity dispersion at core bounce and in the post-collapse phase. Figs.7 (a) and (b) are relatively typical of all our results in the inner and outer parts of the cluster, respectively. It is especially important in late core collapse to exclude the “internal” kinetic energy of binaries, especially hard ones, and in our computations this was done as follows. For regularised binaries only the velocities of the centres of mass contributed to the average velocity dispersions between successive Lagrangian radii. For non-regularised binaries, which are typically much softer than regularised ones, the internal (relative) motions of the components were not subtracted. This should not have any serious effects on the determination of the velocity dispersion because these binaries have internal speeds less than or comparable to the velocity dispersion itself.

First we discuss the velocity dispersion beyond the half-mass radius. This agrees quite well with the predictions of an isotropic gas model for the collapse phase, but during the post-collapse phase the disagreement between the models increases with time. The discrepancy first becomes noticeable at about the time of core bounce, and can be associated with the development of anisotropy in the outer parts of the system (see §4.3 below and Giersz & Spurzem 1993). The anisotropy A reaches values of about 1 at time $t \sim 1000$ (for $N = 500$) and fully accounts for the differences between the models at this time. The occasional ‘spikes’ visible in Fig.7b have a different origin: examination of individual cases

shows that the velocity dispersion changes abruptly during passage of energetic escapers. Although the details of this process are suppressed in Fig.7(b), which averages results over large numbers of cases, we can still see a few spikes which are associated with very energetic products of interactions between stars and binaries.

For the gas model shown in Fig.7(b) the results are insensitive to the treatment of binaries, but this becomes relatively more important in discussion of the inner shells of the models. This is illustrated in Fig.7(a), which shows how the velocity dispersion evolves for stars between Lagrangian radii corresponding to the innermost 1% and 2% of the mass. The isotropic gas model provides satisfactory agreement with the N -body results in the post-collapse phase, at least well after core bounce. At around the time of core bounce gas models with stochastic binary heating provide quite satisfactory fits to the N -body data, but gas models with continuous energy sources give too low a velocity dispersion. This confirms the conclusion of §3.2 that stochastic effects are very important during core bounce. (The gas model whose results are plotted in Fig.7a incorporates steady heating which is, however, only included after $t = 130$, cf. Giersz & Spurzem (1993). For the value of C_b see §3.2).

The temporary episodes of collapse and reexpansion of the core, referred to in §3.2, also contribute a signature, though of smaller amplitude, in the central velocity dispersion, when this is plotted for individual cases.

4.2 Evolution of the Velocity Profile

An alternative view of the evolution of the velocity dispersion is exhibited in Fig.8, which shows velocity dispersion profiles at various times. Since our data is relatively crudely binned by radius these profiles have relatively little detail, especially in the outer zones. Within the inner zones, on the other hand, where the number of stars is smaller, the results are subject to greater statistical uncertainty, and so these plots were generated by averaging over 5 successive times. The mean square speed within a Lagrangian shell is scaled by the central value of the mean square velocity and plotted at the mean radius of the shell, scaled by the half-mass radius.

Despite the crudeness of these profiles, what is clear is how closely self-similar they appear to be in the post-collapse phase of the evolution. Indeed the lines for post-collapse evolution are almost indistinguishable, except for fluctuations.

The technique of computing the logarithmic density gradient (§3.3) can be extended to allow the computation of the profile of the logarithmic velocity dispersion gradient. The results are rather similar to the corresponding profiles of gaseous models, except in the outer parts of the system (i.e. radii beyond the half-mass radius). During the post-collapse evolution the N -body models develop a somewhat smaller gradient of velocity dispersion than gas models do. This may be connected with the development of anisotropy in the N -body models, although anisotropic gas models of Spurzem (1993) also failed to predict this feature. However, as was stressed by Giersz & Spurzem (1993), anisotropic gas models also failed to reproduce the correct *spatial* evolution of the N -body models at the outermost Lagrangian radii. Possible explanations for these discrepancies include the non-local nature of heat transfer in N -body systems, which is connected with close two-body interactions and interactions with hard binaries. These affect the distributions of density and velocity dispersion at locations far from the site of the interactions, and so cannot be modelled

properly by anisotropic gas models with local heating (cf. Giersz & Spurzem 1993).

4.3 Anisotropy

As a measure of the anisotropy we have adopted the definition $A = 2 - 2\sigma_t^2/\sigma_r^2$, where σ_t and σ_r are the one-dimensional tangential and radial velocity dispersions, respectively. Each was computed as a mass-weighted average taken over all stars within spherical shells bounded by consecutive Lagrangian radii.

There is no discernible anisotropy in the innermost shells up to the Lagrangian radius for 10% of the mass. For intermediate shells the anisotropy starts to increase nearly at the time when binaries start to influence the core evolution. For these shells it then increases nearly linearly with time, and eventually reaches a maximum value at about the time when the cluster enters the phase of homologous expansion. Finally, for the outermost shells the anisotropy increases nearly linearly from the very beginning of the calculation, and eventually reaches a maximum value at about the same time as for the intermediate shells. The maximum value of the anisotropy depends on the position within the cluster and on the total number of stars. It increases markedly with radius, and it seems that the maximum value is smaller for larger systems (particularly for the outermost shells).

Our results suggest that some aspects of anisotropy are closely connected with binary activity, especially for small N : interactions of binaries with single stars, and the expulsion of stars and binaries from the core to the outer parts of the system. This conclusion is supported by the facts that the anisotropy starts to increase in intermediate shells at about the same time as binaries start to influence the core evolution, and that it levels off at about the same time as the energy generated by binaries has become adjusted to the demands of the overall cluster expansion. A detailed discussion of all aspects of anisotropy, and a comparison between N -body data and anisotropic gaseous models, are presented in Giersz & Spurzem (1993).

5. CORE EVOLUTION

To some extent the basic data of core evolution have already been covered in §§3.2 and 4, and even parts of §2. But it is such an important feature of star cluster evolution that it deserves an integrated study, in which all this data is synthesised.

5.1 Core Bounce

Of the various quantities we studied, we have found that measurement of the central potential provides what is apparently the sharpest estimate of the time at which core collapse may be considered to end (Fig.9). In fact the central potential can itself be estimated in several ways, of which we have considered two: the potential at the location of the density centre (defined in Casertano & Hut 1985), and the minimum (over all stars in the cluster) of the potential at the location of each star; though in this case the contribution of the nearest neighbour of each star is omitted, in order to exclude the contribution from a binary companion. On average the former measure gives slightly lower values for the central potential, and it is slightly noisier than the latter.

These results show rather clearly that core collapse ends relatively later (i.e. when the collapse is more advanced) in the larger models. Thus the time of core bounce, t_{coll} , is not simply proportional to $N/\ln(\gamma N)$, as would be the case if it were simply proportional

to the half-mass relaxation time. In fact we have found that t_{coll}/t_{rh} is approximately 13.5, 14.1, 16.8 and 17.3 for $N = 250, 500, 1000$ and 2000, respectively, where t_{rh} is the initial half-mass relaxation time (Spitzer 1987, p.40, except that the value of γ has been taken as 0.11, from Paper I, instead of Spitzer's value of 0.4). The depth of the potential well at core bounce is also greater for larger N .

Incidentally the data for $N = 2000$ show two minima of rather similar depth at times $t \simeq 330$ and 360 in Fig.9, but our other measure of central potential (see above) clearly resolves the issue and implies that the later is the true minimum. Another detail worth recording is the broad range of times of core bounce exhibited by different cases with the same N . Actually this is quite difficult to determine for $N \leq 500$, but for $N \geq 1000$ the range of values has a length about 50% of the mean value. To the extent that the value can be found for smaller N the absolute range is somewhat smaller, but comparable with the mean time of core bounce. It seems that the range in the time of core bounce, expressed as a fraction of the mean time of core bounce, decreases with increasing N , as is certainly plausible on the simplest grounds. Unfortunately, however, we cannot quantify this statement because of substantial uncertainties in the determination of the time of core bounce for systems with $N \leq 500$.

Theoretically, conditions at core bounce are usually estimated by assuming that the energy emitted by binaries on the time scale of the collapse is comparable with the energy of the core (Heggie 1984, Goodman 1987). This leads to the condition that

$$N_{cb} \propto \left(\frac{f|\phi_c|}{\sigma_c^2 \xi \ln \Lambda} \right)^{1/2} \quad (17)$$

(cf. eq.(12)), where ξ is the dimensionless core collapse rate (Cohn 1980), f is a coefficient of order 1 which determines the efficiency of the energy sources, N_{cb} is the number of stars in the core at core bounce, $\ln \Lambda$ is the Coulomb logarithm, and it is assumed that the main energy generation mechanism is associated with the formation and evolution of binaries in three-body encounters. Assuming that the core density and radius (ρ_c and r_c) are related by $\rho_c \propto r_c^{-\alpha}$, as in the standard theory of core collapse (Lynden-Bell & Eggleton 1980), this leads to the condition (at core bounce) that

$$\rho_c \propto \left(\frac{\xi N^2 \ln \Lambda}{f|\phi_c|/\sigma_c^2} \right)^{\frac{\alpha}{2(3-\alpha)}}, \quad (18)$$

where as usual we use standard units (in which the total initial mass of a system is unity). The results of Fig.10 and Table 4, however, indicate that the central density at the time of core bounce is not a simple power of N within the range we have studied. Within the context of the above theory, this indicates that one or more of the parameters in eq.(18) are variable. Indeed Fokker-Planck models indicate that α and ξ do evolve as the core collapses (Cohn 1980), and we have already seen (Fig.9) that the value of ϕ_c at the time of core bounce depends on N . Indeed the character of late core collapse already looks different for $N = 1000$ and 2000 in Fig.10 than in the results of the other two sets of models.

The theoretical interpretation of the density at core bounce is actually somewhat more involved than the above discussion suggests. One complication, already mentioned in §3.2, is the effect of averaging over many cases in which the core bounce can occur at very different times. It can be easily appreciated that this tends to suppress the mean density at its peak (at core bounce). This variation of bounce times is also exhibited in gas models with stochastic binary formation and evolution, but not to the same extent.

What has been said above about the central density at core bounce applies also, with appropriate changes, to the central velocity dispersion. The N -dependence of the maximum central root mean square velocity is much milder, the value for $N = 2000$ being only about 18% larger than for $N = 250$ (Fig.11). Again the most successful model for binary heating is one including both stochastic formation and stochastic burning.

We have also determined the core radii in our models according to the following prescription: Lagrangian radii for the one or two innermost shells (for $N \leq 500$ we used two shells and for $N > 500$ only one), together with the average velocity dispersion in these shells, were used to estimate the central density and velocity dispersion. The core radius was then calculated using the standard definition $r_c^2 = 9\sigma_c^2/4\pi G\rho_c$. Because of the relatively modest time-dependence of the velocity dispersion, the behaviour of the core radius is almost entirely determined by that of the central density. So, again, the evolution is best described by a gas model including stochastic formation and stochastic burning of binaries. The main core parameters at the time of core bounce are presented in Table 4.

A relatively direct test of the theoretical argument leading to eq.(17) is in the sixth column of Table 4. It shows that the number of stars in the core at core bounce (i.e. the number within radius r_c) is not independent of N , as often assumed, but increases with increasing N , at least within the range we have studied. The trend is consistent with eq.(17) in the sense that, for larger N , collapse would be expected to proceed further (e.g. in terms of central density); and it is known that ξ decreases and $|\phi_c|/\sigma_c^2$ increases as core collapse deepens (Cohn 1980).

5.2 Post-Collapse Evolution of the Core

For the purpose of theoretical comparison the simplest choice is the series of gas models computed by Goodman (1987), which assume a simple model of energy generation by binaries, and self-similar evolution. Goodman's models are stable for the range of values of N considered here. The inner Lagrangian radii in our N -body results, at least up to the 75% radius (Fig.4), approximately confirm the homologous nature of the evolution. Other gas models, without the assumption of self-similarity, have also been computed in order to determine the effects of different assumptions on the rate and character of energy generation by binaries.

The actual value of the central density is quite sensitive to the assumed rate of energy generation by binaries, and what was said about eq.(13) in the context of the inner Lagrangian radii in §3.2 is applicable here; in other words this formula, with a suitably adjusted numerical coefficient, provides a good fit in the post-collapse regime, though inclusion of stochastic effects is needed for a satisfactory fit at core bounce. Similar remarks may be made about the central velocity dispersion. Indeed, when the time units are adequately scaled (Fig.11), the differences (for different N) which appear at around the time of core bounce rapidly diminish.

Since the core radius depends solely on the central density and velocity dispersion, we also find that its post-collapse evolution (substantially after core bounce) fails to discriminate the models with stochastic binary formation and burning from those with smooth burning. There is, however, one interesting feature which may indicate the role of stochastic binary evolution, which is a slight dip in the otherwise monotonic increase in the core radius. For $N = 250$ it occurs at about $t \simeq 160$; it may also be glimpsed in Fig.10 as a slight rise at the scaled time of $t \sim 500$ for $N = 250$. It is possible that this marks a tendency for recollapse of the core at a time when the first energetic binary is ejected from the core. This time will vary greatly from one case to another, but the signal of this process may appear weakly when many cases are averaged. A similar feature may be present in our results of models incorporating stochastic burning and formation of binaries, but in any case is somewhat elusive. What is not in question is the evidence for this recollapse in individual cases (e.g. Fig.5).

In the homologous phase after core collapse, the N -dependence of the core radius qualitatively follows theoretical predictions, in the sense that, for larger N , it is a smaller fraction of the half-mass radius (see Fig.12). The number of stars in the core in the early post-collapse phase tends to stay fairly close to the value at the end of core collapse (Fig.13), and so the N -dependence of N_c at that time approximately persists for a time comparable with the collapse time itself; and indeed for a further 11 and 6 collapse times for $N = 250$ and 500, respectively. We did not follow the evolution of the cases with $N = 1000$ for so long, relative to the collapse time, and so the long-term evolution of N_c is unclear in this case. From Figs.12 and 13, however, it seems that core bounce has a visible influence for larger N : the ratio of r_c/r_h and the number of stars in the core at the time of core bounce are slightly smaller than during the post-collapse phase, for $N = 2000$. This is in agreement with theoretical predictions, which, in the limit of large N , predict that the value of N_c at core bounce should be almost independent of N (eq.17), whereas in homologous post-collapse expansion it varies as $N^{1/3}$ (eq.8). For models with lower N this feature is not present, because the core collapse does not proceed far enough and, as we saw in §5.1, the N -dependence of eq.(17) is complicated by other factors. The number of stars in the core predicted by Goodman's gas model (Goodman 1987) is rather too large for $N \leq 1000$, but for $N = 2000$ it is very close to that found in the N -body calculations.

5.3 The Evolutionary Track of the Core

The foregoing discussions have considered the time-dependence of various quantities, but it is also of interest to consider the variation of two (or more) fundamental core parameters against each other. For example, in the context of gas and Fokker-Planck models this has proved to be a useful diagnostic for the study of oscillatory post-collapse evolution (Goodman 1987, Breeden *et al* 1990).

Fig.14 shows results for $N = 500$ and a gas model with continuous energy generation at a rate which has been found empirically to give satisfactory agreement with core parameters in post-collapse evolution (see §3.2). As we have already observed, this model does not produce a sufficiently deep collapse, but the post-collapse evolution of the core is followed quite closely.

The post-collapse behaviour shown in Fig.14 is reasonably well explained by the usual theory of homologous post-collapse evolution. If we suppose that the velocity dispersion

varies as $v_c^2 \propto GM/R$, where M is the total mass and R is a scale radius, and that the core density varies as $\rho_c \propto M/R^3$, then we find that

$$\frac{d \ln v_c^2}{d \ln \rho_c} \simeq \frac{1}{3} \left(1 - \frac{2}{3} \frac{d \ln M}{d \ln R} \right). \quad (19)$$

Since we find (from eqs.(14) and (15) and Table 3) that $d \ln M / d \ln R \simeq -0.128$ for $N = 500$ it follows that the expected slope in the N -body data should be about 8% steeper than in the gas model (in which, by assumption, no mass is lost). This can be confirmed qualitatively in Fig.14.

6. ESCAPE

6.1 Rate of Escape

In homologous post-collapse evolution it is easy to show that the rate of escape in two-body encounters should give rise to a power-law dependence of the total mass on time, as in eq.(14) for some value of ν . Note that this mass loss is not included in the detailed theoretical post-collapse models which we have used up to now, i.e. those of Hénon (1965) and Goodman (1987). Goodman (1984) does indeed discuss and include mass-loss, but it is the mass loss which is associated with the mechanism for powering the post-collapse expansion.

We find from the results of our N -body models that eq.(14) fits all models with $250 \leq N \leq 2000$ with values of ν given in Table 3. A suitable average value for the power index is $\nu = 0.086$. These results are suitable for theoretical purposes, but do not give a good intuitive feel for how quickly clusters lose mass. Fig.15 therefore gives the raw data.

From what was said in Paper I about the changing rate of escape during core collapse, two factors contribute to the success of eq.(14): one is that the core evolves nearly homologically with the half-mass radius, and the other is the near-constancy of the anisotropy in the post-collapse phase (as far as we have followed this). As in Paper I we have attempted to reproduce the observed rate of escape by computing the rate of escape in an isotropic Fokker-Planck model using Hénon's general formula (eq.(2) in Paper I), and correcting for anisotropy in the very approximate way discussed in Paper I (§3.3.3). This is fairly successful for all N , but the goodness of fit depends on the assumed escape radius for N -body models (see discussion in Paper I).

6.2 Energy of Escapers

Though the theory just described is partially successful in accounting for the number of escapers, it grossly underestimates the energy they carry off in the post-collapse regime. (Here we refer to the translational kinetic energy of single stars and the barycentres of escaping binaries and other bound subsystems.) The actual data are given in Fig.16, and immediately after the end of core collapse the values increase much more sharply than predicted by a theory based on escape by two-body interactions.

Fig.17 helps to explain in what way the theory of two-body escape, which accounts quite well for the fluxes of both mass and energy during core collapse, fails in the post-collapse phase. From the end of core collapse onwards, a new class of escapers of much

higher energy occurs. Their numbers are relatively small, but their energy is higher by about two orders of magnitude, and they dominate the energy flux. It is natural to associate them with three-body interactions, involving the hardening of binaries. A histogram of the energies of all escapers (from time $t = 0$ until the end of the calculations) shows a bimodal form, but it is clear that the distribution is time-dependent.

To clarify this point we have carried out a Monte-Carlo simulation of interactions between binaries and single stars using Spitzer's and Heggie's scattering cross sections (see §2.4). Briefly, a binary is created with an initial binding energy $\varepsilon \geq 3kT$ distributed according to $f(\varepsilon) \propto \varepsilon^{-9/2}$ (Heggie 1975), and allowed to evolve according to the appropriate cross section. Assuming that the central potential (scaled by the central one-dimensional velocity dispersion) is constant (see Fig.20) the escape energies of all escapers are recorded, until the binary itself escapes. The number of trials computed was of order 10000, which is about 20 times larger than the total number of hard binaries in the N -body systems. We have found that the maximum escape energy is as large as $1500kT$, while the minimum escape energy is as small as $10^{-4}kT$. Both extreme values depend only slightly on the assumed scattering cross section and scaled central potential, and they are in satisfactory agreement with our experimental data, when allowance is made for the number of trials.

The presence of these escaping particles gives rise to a small virial imbalance. For $N = 250$, for example, the virial ratio (defined as the ratio of the external kinetic to external potential energy) remains within the range $0.50 - 0.51$ during core collapse, but then rises abruptly over the next collapse time. Thereafter the average value remains close to about 0.55, though with increased fluctuations.

6.3 Binary Escapers

Associated with the escape of the most energetic single escapers are escaping binaries (Fig.18). This diagram illustrates particularly well the value of averaging results from many simulations: such small numbers are involved that the results from an individual case are dominated by the stochastic nature of binary evolution. The numbers of these escapers are negligible compared with the total numbers of escaping stars (Fig.15), which are therefore completely dominated by single escapers.

Using this data, along with information on the evolution of the half-mass radius or the external energy of the bound cluster, we can test one of the basic assumptions of simple theory, in which the escape of each binary is assumed to be accompanied by the release of an amount of energy which is a given multiple of the mean individual kinetic energy of the bound stars (Cohn 1985, Goodman 1984). This assumption implies that $d \ln |E_{ext}| / dN_{besc}$ is constant for each N , where N_{besc} is the number of binary escapers and E_{ext} is the external energy of the bound cluster. We found values: -0.69 , -0.46 , -0.32 and -0.21 for $N = 250$, 500 , 1000 and 2000 , respectively. If, as is often assumed, each binary releases a fixed multiple of the mean kinetic energy of a single star, the result would be proportional to N^{-1} . In fact the N -dependence is complicated by the following fact. For larger N , core collapse proceeds further, and the value of the central potential (scaled by either the global or central value of kT) is larger (during homologous expansion) than for smaller N (cf. Fig.20). Hence, on average, binaries reach larger energies (scaled by kT) before escaping (cf. Fig.2).

Though the *number* of escaping binaries is relatively insignificant, their external energy

(i.e. the translational kinetic energy associated with their barycentric motion) is a larger fraction of the energy of all escapers, but even so it is smaller than the latter by a factor of about 3.5. Where the escaping binaries are energetically very important is in their *internal* energy (Fig.19), which becomes comparable with the internal energy of *bound* pairs quite early in the post-collapse phase.

An especially interesting parameter, related to the internal and external energy and number of escaping binaries, is the *mean* change of the internal binding energy of a binary in the encounter which leads to its escape, as this is theoretically related to the amount of energy which each binary may donate to the cluster.

The theoretical argument goes as follows. Assume that a binary of internal binding energy ε experiences a three-body encounter in which its binding energy increases by $\delta\varepsilon$ to ε' . If the change in energy much exceeds the rms speeds of the stars, then the energy given to the barycentric motion of the binary is approximately $\delta\varepsilon/3$. Therefore the binary escapes if $\delta\varepsilon/3 > mv_{esc}^2 - mv_c^2/2$, where m is the mass of a single star or binary component, v_{esc} is the escape speed and v_c is the central velocity dispersion; (we assume equipartition in the core). By the same argument we deduce that the relative change of energy of a binary, in an encounter leading to its escape, satisfies

$$\Delta = \frac{\delta\varepsilon}{\varepsilon} = \frac{3 \left(x_{ex} + 2 \frac{|\phi_c|}{\sigma_c^2} - 1.5 \right)}{x_{in} - 3 \left(x_{ex} + 2 \frac{|\phi_c|}{\sigma_c^2} - 1.5 \right)}, \quad (20)$$

where x_{ex} and x_{in} are the external and internal energy of the escaping binary, respectively, both expressed in units of $1kT_c$ ($\equiv m\langle v_c^2 \rangle/3 \equiv m\sigma_c^2$). For the mean value of the ratio Δ the value of 0.4 is frequently quoted, based on certain analytical estimates (Heggie 1975) which are restricted to close encounters, or numerical estimates (Hills 1975) which are restricted to zero impact parameter. In fact inclusion of wide encounters reduces the average, and indeed strictly it vanishes, because of the arbitrarily large number of distant encounters which have a negligible effect on ε . Nevertheless in eq.(20) we are concerned only with encounters which lead to the escape of binaries, and so we can expect that distant encounters contribute only slightly.

In order to check this formula we require some further information, especially concerning the escape speed. For this purpose, we show in Fig.20 the central potential in units of the central one-dimensional velocity dispersion. There is remarkable consistency during the core collapse phase, but in this paper our emphasis is on post-collapse evolution, where there is clear evidence of an N -dependence, with larger values for larger N , though the dependence is no faster than about $N^{1/6}$. This dependence presumably reflects the fact that the ratio of core to half-mass radius decreases as N increases (cf. §5.2).

Because of the size of our sampling interval, only for a fraction of all binary escapers were we able to trace back to determine the value of Δ *directly*, i.e. by computing Δ from the definition $\Delta = (\varepsilon' - \varepsilon)/\varepsilon$. For all other cases eq.(20) was used to *estimate* the value of Δ .

Examination of our data shows that the relative change of the binary internal energy, in those encounters which lead to the escape of the binary, has a very asymmetric distribution, with a very long tail for large Δ . The distribution is peaked at Δ around 0.5, and the

average value of Δ is about 1.0 (cf. Fig.21). As can be seen in Fig.21, the “direct” and estimated distributions are very similar, and so it appears that eq.(20) gives quite a good approximation for Δ . However the distribution of estimated values is slightly too high for large Δ . A possible explanation for this difference could be the fact that we approximate the central velocity dispersion at the time of the encounter which leads to the escape of the binary by the value at the time when the binary is removed from the system. This velocity should be generally smaller than at the time of the encounter, since the escape often leads to a modest expansion of the core, and so our estimates are shifted towards larger values. Other possible explanations are mentioned below, along with a discussion of the internal energies of the escaping binaries.

As we mentioned in §6.2 we conducted Monte-Carlo simulations for the evolution and escape of binaries, using two different theoretical scattering cross sections (see §2.4). For both cross sections the calculated distributions of Δ (for the final encounters leading to escape) have a very long tail similar to those of the N -body data in Fig.21. The distribution of Δ obtained for Spitzer’s formula drops more steeply than that for Heggie’s formula for $\Delta \geq 1$, and is peaked at around 0.5, compared with a value of 0.7 for Heggie’s cross section. Both formulae give negligible numbers of interactions with $\Delta \leq 0.2$ which is in contradiction with our experimental findings. The average values of Δ for Spitzer’s and Heggie’s formulae are about 1.09 and 1.33, respectively. We can conclude that Spitzer’s formula gives closer results to those obtained from our full N -body simulations. Unfortunately we cannot quantify this statement satisfactorily, because of the limited statistics of binary escapers from N -body models.

The internal energies of escaping binaries, when expressed in terms of the central velocity dispersion, have a distribution with a very long tail up to around $2000kT$, and about 9% of all escaping binaries have energies $\leq 100kT$. The distributions are peaked at around 125, 175 and 225 kT for $N = 250, 500$ and 1000, respectively. The theoretical distributions again fail to predict the number of binary escapers with internal energies $\leq 100kT$, as they give values which are several times too small. However the shapes of all distributions, both experimental and theoretical, are very similar for internal energies $\geq 200kT$, and the theoretical distributions are peaked at nearly the same values as the experimental ones. The distributions for Spitzer’s and Heggie’s formulae are practically the same.

There are at least two possible explanations for the differences between the theoretical and experimental distributions of the internal energy of binary escapers for $\varepsilon' \leq 100kT$. (These may also contribute to the differences, mentioned above, in the theoretical and experimental distributions of Δ for $\Delta \leq 0.2$.) One possible explanation is connected with the fact that a few *hierarchical binaries* escape from the systems, i.e. binaries accompanied by a distant but bound third body. In these cases our data may refer to the “outer” binary. Another possible explanation involves the escape of relatively soft binaries on very elongated orbits by ordinary two body processes (Stodólkiewicz 1985). It is possible that the left hand parts of the experimental distributions in Fig.21 are contaminated by one or both of these processes.

Despite the large fluctuations in the experimental distributions for ε' and Δ they resemble the theoretical distributions obtained from Spitzer’s scattering cross section more

closely than those from Heggie’s cross section. Another indirect piece of evidence supporting this conclusion can be obtained from a study of the total internal energy of *bound* hard binaries. If we assume that this energy is mainly contributed by one very hard binary (cf. §2.3), it may be estimated from our data that its energy exceeds $250kT_c$ on average for all N , in the post-collapse phase. The theoretical estimates give values of about $220kT_c$ and $120kT_c$ for Spitzer’s and Heggie’s scattering cross sections, respectively. Thus it seems again that Spitzer’s formula gives better agreement with the experimental data than Heggie’s one.

7. DISCUSSION AND CONCLUSIONS

This paper has been concerned with the evolution of isolated systems of bodies of equal mass, in those phases of the evolution in which hard binaries play an important role. It is unfortunate, then, that one quantity on which our data is vague is the *number* of such binaries (§2.2), as we instead have data on the number of pairs which are “regularised” in the N -body simulations. Though the difference between these two numbers may correspond to pairs which are on the threshold between hard and soft, and may be dynamically unimportant, this flaw in our methodology should be remedied.

The other aspect of our data, as presented here, which is extremely limited is the dispersion of the results, i.e. the size of the differences between different cases at the same time. This is of special interest in the context of the post-collapse expansion of the core (Fig.5), which in individual cases shows alternating phases of expansion and recollapse (§3.2). The relation between these types of behaviour and the evolution of binaries is far from clear. The naive expectation that escape of binaries causes core expansion is too simplistic, because there are energetic interactions which do not appear to initiate expansions, and some expansions begin with interactions which do not seem particularly energetic.

These issues apart, our technique of combining data from relatively large numbers of N -body simulations has allowed us to investigate a number of fundamental dynamical processes with data of relatively high statistical quality. Concerning interactions of hard binaries, several lines of evidence support the applicability of the simple scattering cross section presented by Spitzer (1987). These include (i) study of the changes in the internal energy of hard binaries (§2.4), (ii) the evolution of the core (§3.2); and (iii) the internal energies of escaping binaries (§6.3). The effect of these binaries on the core, however, requires some attention to the following points (§§3.2, 5.1): (i) models in which energy is generated smoothly, i.e. at a given average rate, do not account well for the depth of core collapse, and the role of fluctuations is also indicated by a study of the formation rate of hard binaries during core collapse (§2.1); (ii) even in the post-collapse phase the generation of energy by these processes seems less efficient than is usually assumed, at least for smaller N (§3.2).

In the post-collapse phase the evolution is remarkably self-similar, whether one looks at the density gradient (§3.3) or the profile of the dispersion of velocities or anisotropy (§§4.2, 4.3). The number of stars in the core remains nearly constant, though it depends on N (§§5.1, 5.2). Comparison of the evolution of the half-mass radius with simplified models allow a redetermination of the values of the thermal conductivity coefficient (for

gas models) and the argument of the Coulomb logarithm in the expression for the relaxation time. The former, expressed in terms of the value of C (Lynden-Bell & Eggleton 1980), gives a best-fitting value close to 0.164 (§3.1), which is about 50% larger than the value determined in Paper I from the evolution of the inner radii during core collapse. Similarly, if the argument of the Coulomb logarithm is expressed as γN , the best value of γ is about 0.035, less than half the value found in Paper I.

The power-law expansion of the half-mass radius is matched by the time-dependence of the total bound mass; this varies nearly as $(t - t_0)^{-\nu}$, where a fairly consistent value $\nu \simeq 0.086$ is indicated for all the values of N we have studied (Table 3). Most escapers result from two-body encounters, but from about the time of core bounce they are joined by a small number of escapers, both single stars and binaries, which are attributable to interactions involving binaries. They are clearly distinguishable by their energies (§6.2).

It can be seen that N -body data of suitable statistical quality can be very useful for “calibrating” various dynamical processes of importance in the evolution of stellar systems, for example, binary heating. Such data also demonstrate the limitations of the commonly made assumption that binaries contribute energy continuously to the core rather than stochastically. N -body models also yield data on important aspects of stellar dynamics which are difficult to study by simplified techniques, e.g. stellar escape. These simplified techniques remain, however, the ones which are most directly applicable to systems such as globular star clusters, because of the relatively small values of N to which useful N -body models are restricted. Eventually N -body models will become the method of choice for studying the dynamical evolution of globular star clusters, but for the next decade at least reliance must be placed on simplified models, and for the time being one of the important tasks of N -body modelling is to check the assumptions and parameters underlying these simplified methods.

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Table 1
NUMBER OF MODELS AT THE END OF CALCULATIONS

N	N_{ci}	N_{ce}	t_{end}
250	56	50	1000
500	56	48	1000
1000	50	41	600
2000	16	15	2300

Note: N_{ci} and N_{ce} are the initial and final numbers of computed models, and t_{end} is the final time in N -body units.

Table 2
BINARY FORMATION DURING CORE COLLAPSE

N	250	500	1000	2000
t_1	47	119	262	500
t_3	81	135	284	550
t_{10}	87	151	300	555
t_{coll}	95	164	332	600

Note: t_1 is the median time (over a number of simulations) at which a binary with energy exceeding $1kT$ first appears. The quantities t_3 and t_{10} are similarly defined for energies of $3kT$ and $10kT$ respectively. The quantity t_{coll} is an estimate of the mean time of the end of core collapse (cf. §5.1).

Table 3
MASS LOSS AND RELAXATION PARAMETERS

N	ν	t_{min}	t_{max}	t_0	γ_1	C_1	γ_2	C_2
250	0.083	360	1000	-8.3	0.054	0.142	0.041	0.160
500	0.089	400	1000	-9.1	0.040	0.143	0.030	0.158
1000	0.082	420	600	19.4	0.045	0.142	0.035	0.152
2000	0.090	700	2300	-45.6	0.067	0.147	0.044	0.162

Notes: The parameter ν is defined in the discussion of eqs.(14) and (15). This and other parameters listed have been determined over the range of times (t_{min}, t_{max}). The parameter t_0 is the origin of time which permits the best fit of eqs.(14) and (15) to the N -body data. The parameters γ and C are, respectively, the coefficient of N in the argument of the Coulomb logarithm, and a coefficient in the expression for the conductivity in the gas model. Subscripts 1 and 2 are explained in §3.1.

Table 4
CONDITIONS AT CORE BOUNCE

N	t_{coll}	ρ_c	v_c	r_c	N_c
250	95	22.5	0.93	0.098	12.6
500	164	42.3	0.97	0.069	20.0
1000	332	125.9	1.04	0.047	28.2
2000	600	281.2	1.10	0.034	44.7

Note: ρ_c , v_c , r_c and N_c are, respectively, the central density, the rms three-dimensional speed of stars in the core, the core radius, and the number of stars within this radius, all measured at the time of core bounce, t_{coll} .

FIGURE CAPTIONS

Fig.1 Number of regularised binaries. The time units for $N = 250$, 500 and 2000 have been scaled to that for $N = 1000$ according to results of Paper I. Only binaries which are not escapers (§6.3) are included.

Fig.2 Maximum binding energy of bound regularised binaries. The time units have been scaled as in Fig.1. Recall that the initial binding energy of each system is $-1/4$.

Fig.3 Total “external” binding energy of all bound members. The time units have been scaled as in Fig.1.

Fig.4 Lagrangian and core radii for $N = 500$.

Fig.5 Evolution of the inner Lagrangian radii for one 250-body model (case 25). The most energetic interactions of binaries are marked at the lower border, thick lines indicating binary escapers.

Fig.6 Logarithmic gradient profiles of density at several times (identified in the key). The radii are scaled by the half mass radius. a) N -body models for $N = 250$, b) N -body models for $N = 2000$, c) isotropic gas model for $N = 250$, d) isotropic gas model for $N = 2000$. For the gas models C_b is the value of a coefficient in the binary heating formula, eq.(13), and t_{b0} is the time at which this is switched on (cf. Giersz & Spurzem 1993). For the N -body models data on Lagrangian radii have first been smoothed over an interval of order (a) 20 and (b) 70.

Fig.7 Evolution of the three-dimensional velocity dispersion with time (a) between the Lagrangian radii corresponding to 1 and 2% of the mass, and (b) between the Lagrangian radii corresponding to 50 and 75% of the mass, for models with $N = 500$. The dashed line gives the corresponding result for an isotropic gas model with parameters for energy generation given by $C_b = 70$ and $t_{b0} = 130$ (cf. the caption to Fig.6).

Fig.8 Velocity dispersion profiles at several times throughout the pre- and post-collapse evolution of systems with $N = 500$. The key identifies the N -body time corresponding to each curve. The radius is scaled by r_h , and the velocity dispersion by the central value.

Fig.9 Central potential (potential at the position of the density centre) for models with $N = 250, 500, 1000$ and 2000. The time units have been scaled as in Fig.1.

Fig.10 Central number density for models with $N = 250, 500, 1000$ and 2000. The time units have been scaled as in Fig.1.

Fig.11 Central root-mean-square three-dimensional velocity for models with $N = 250, 500, 1000$ and 2000. The times have been scaled as in Fig.1.

Fig.12 Ratio between core and half-mass radii for $N = 250, 500, 1000$ and 2000. The times have been scaled as in Fig.1.

Fig.13 Number of stars in the core (i.e. the number within core radius r_c) for $N = 250, 500, 1000$ and 2000. The times have been scaled as in Fig.1.

Fig.14 Evolution of core parameters ρ_c and v_c^2 for $N = 500$ and (dashed curve) for an isotropic gas model specified as in Fig.7. The N -body data were smoothed using the standard routine SMOOFT from Press *et al* (1986).

Fig.15 Number of escaping stars for models with $N = 250, 500, 1000$ and 2000.

Fig.16 Dependence of “external” energy of escapers on time for $N = 250, 500, 1000$ and 2000. Spikes observed for $N = 250$ are connected with temporary loss of data on individual models due to hardware problems.

Fig.17 Scatter diagram showing the external energies (scaled by the central value of kT) of all single escapers in models with $N = 1000$. Very similar results are obtained for other values of N .

Fig.18 Number of escaping binaries for models with $N = 250, 500, 1000$ and 2000 .

Fig.19 Internal energy of escaping binaries for models with $N = 250, 500, 1000$ and 2000 .

Fig.20 Central potential in units of the central one-dimensional velocity dispersion. The data have been smoothed as in Fig.14, and the times have been scaled as in Fig.1. For a Plummer model ($t = 0$) the theoretical value is $3\phi_c/v_c^2 = 6$.

Fig.21 Histograms of the last relative energy change, Δ , for escaping binaries. Thick line - data directly obtained from N -body models; thin line - data estimated from eq.(20). The total number of binaries in each category of data is N_t , and the ordinate is the fraction of cases falling within the corresponding bin.